# Part II - Normally-Distributed Portfolio Value

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September, 2018

In Part I we built a model for asset value that assumed that asset values were normally-distributed. In Part II we will build a model for portfolio value that assumes that individual asset values are normally-distributed and correlated. To that end we will use the following hypothetical problem...

## Our Hypothetical Problem

We are tasked with building an portfolio value model that assumes that asset prices are normally-distributed. We are given the following go-forward model assumptions...

**Table 1: Portfolio Composition**

| Description | Asset 1 | Asset 2 | Asset 3 | Total |
|---|---|---|---|---|
| Asset value at time zero (in dollars) | 300,000 | 500,000 | 200,000 | 1,000,000 |
| Annual return mean | 12.00% | 10.00% | 8.00% | – |
| Annual return volatility | 30.00% | 20.00% | 10.00% | – |
| Annual distribution rate | 5.00% | 4.00% | 3.00% | – |

The asset return correlation matrix is...

**Table 2: Correlation Matrix**

| | Asset 1 | Asset 2 | Asset 3 |
|---|---|---|---|
| Asset 1 | 1.00 | 0.60 | 0.40 |
| Asset 2 | 0.60 | 1.00 | 0.50 |
| Asset 3 | 0.40 | 0.50 | 1.00 |

Using the parameters in Table 1 above our model parameters are...

**Table 3: Model Parameters**

| Symbol | Asset 1 | Asset 2 | Asset 3 | Notes |
|---|---|---|---|---|
| $A_i(0)$ | 300,000 | 500,000 | 200,000 | Value in dollars |
| $w_i$ | 0.3000 | 0.5000 | 0.2000 | Asset weight in portfolio |
| $m_i$ | 0.0700 | 0.0600 | 0.0500 | Annual return mean minus annual distribution rate |
| $v_i$ | 0.0900 | 0.0400 | 0.0100 | Annual return volatiity squared |
| $\rho_i$ | 0.6928 | 0.8660 | 0.5774 | See Gaussian Copula pdf [3] |

Our task is to answer the following questions:

**Question 1**: What is portfolio value mean and variance at the end of year three?

**Question 2**: What is portfolio return mean and variance at the end of year three?

**Question 3**: What is the random portfolio value equation?

## Individual Asset Values

We will define the variable $A_i(0)$ to be the i'th asset's value at time zero, the variable $m_i$ to be the i'th assets expected annual rate of return, and the variable $v_i$ to be the i'th asset's expected annual return variance. The equation for asset value at time $t$ that includes correlation is... [1] [2]

$$A_i(t) = A_i(0)\left(1 + m_i\right)^t \left(1 + \sqrt{v_i\,t}\left(\rho_i\,Z_c + \sqrt{1-\rho_i^2}\,Z_i\right)\right) \text{ ...where... } Z_c \sim Z_i \sim N\left[0, 1\right] \tag{1}$$

Using Appendix Equation (33) below the equation for the first moment of the distribution of random asset value at time $t$ is...

$$\mathbb{E}\left[A_i(t)\right] = A_i(0)\left(1 + m_i\right)^t \tag{2}$$

Using Appendix Equation (34) below the equation for the second moment of the distribution of random asset value at time $t$ is...

$$\mathbb{E}\left[A_i(t)^2\right] = A_i(0)^2\left(1 + m_i\right)^{2\,t}\left(1 + v_i\,t\right) \tag{3}$$

Using Equation (2) above the mean of the distribution of random asset value at time $t$ is...

$$\text{mean of } A_i(t) = \mathbb{E}\left[A_i(t)\right] = A_i(0)\left(1 + m_i\right)^t \tag{4}$$

Using Equations (2) and (3) above the variance of the distribution of random asset value at time $t$ is...

$$\text{variance of } A_i(t) = \mathbb{E}\left[A_i(t)^2\right] - \left[\mathbb{E}\left[A_i(t)\right]\right]^2 = A_i(0)^2\left(1 + m_i\right)^{2\,t} v_i\,t \tag{5}$$

Using Appendix Equations (33) and (35) below the covariance of the i'th and j'th asset value at time $t$ is...

$$\text{covar of } A_i(t), A_j(t) \ = \mathbb{E}\left[A_i(t)\,A_j(t)\right] - \mathbb{E}\left[A_i(t)\right]\mathbb{E}\left[A_j(t)\right] = A_i(0)A_j(0)\left(1 + m_i\right)^t\left(1 + m_j\right)^t \rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t \tag{6}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.

## Portfolio Value

We will define the variable $P(t)$ to be random portfolio value at time $t$. Using Equation (1) above the equation for random portfolio value is...

$$P(t) = \sum_{i=1}^{N} A_i(t) \tag{7}$$

We will define the variable $w_i(0)$ to be the i'th asset's weight in the portfolio at time zero. Using Equation (1) above the equation for asset weight is...

$$w_i(0) = \frac{A_i(0)}{P(0)} \text{ ...such that... } A_i(0) = w_i(0)\,P(0) \tag{8}$$

Using Appendix Equation (36) below the equation for the first moment of the distribution of random portfolio value at time $t$ is...

$$\mathbb{E}\left[P(t)\right] = P(0)\sum_{i=1}^{N} w_i(0)\left(1 + m_i\right)^t \tag{9}$$

Using Appendix Equation (38) below the equation for the second moment of the distribution of random portfolio value at time $t$ is...

$$\mathbb{E}\left[P(t)^2\right] = P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\left(1 + m_i\right)^t\left(1 + m_j\right)^t\left(1 + \rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\right) \tag{10}$$

Using Equation (9) above the mean of the distribution of random portfolio value at time $t$ is...

$$\text{mean of } P(t) = \mathbb{E}\Big[P(t)\Big] = P(0)\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t \tag{11}$$

Using Equations (9) and (10) above and Appendix Equation (37) below the variance of the distribution of random portfolio value at time $t$ is...

$$\text{variance of } P(t) = \mathbb{E}\Big[P(t)^2\Big] - \Big[\mathbb{E}\Big[P(t)\Big]\Big]^2 = P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t \rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t \tag{12}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.

We will define the vector $\vec{\mathbf{w}}$ to be a column vector of asset weights. Using Equation (8) above and the model parameters in Table 3 above the equation for this vector is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0.3000 \\ 0.5000 \\ 0.2000 \end{bmatrix} \text{ ...where... } w_i = \frac{A_i(0)}{P(0)} \tag{13}$$

We will define the vector $\vec{\mathbf{r}}$ to be a column vector of asset returns. Using Equation (11) above and the model parameters in Table 3 above the equation for this vector is...

$$\vec{\mathbf{r}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1.2250 \\ 1.1910 \\ 1.1576 \end{bmatrix} \text{ ...where... } r_i = \Big(1+m_i\Big)^3 \tag{14}$$

We will define the matrix $\mathbf{A}$ to be a matrix of covariances. Using Equation (12) above and the model parameters in Table 3 above the equation for this matrix is...

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} 0.4052 & 0.1576 & 0.0511 \\ 0.1576 & 0.1702 & 0.0414 \\ 0.0511 & 0.0414 & 0.0402 \end{bmatrix} \text{ ...where... } a_{i,j} = \Big(1+m_i\Big)^t\Big(1+m_j\Big)^t \rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t \tag{15}$$

Note that using the matrix and vector definitions in Equations (13), (14) and (15) above we can rewrite portfolio value mean and variance Equations (11) and (12) above as...

$$\text{mean of } P(t) = P(0)\,\vec{\mathbf{w}}^T\vec{\mathbf{r}} \text{ ...and... variance of } P(t) = P(0)^2\,\vec{\mathbf{w}}^T\mathbf{A}\vec{\mathbf{w}} \tag{16}$$

## Portfolio Return

We will define the variable $r_p(t)$ to be random portfolio return over the time interval $[0,t]$. Using asset value Equation (7) above the equation for random portfolio return is...

$$r_p(t) = \frac{P(t)}{P(0)} - 1 \text{ ...such that... } P(t) = P(0)\Big(1+r_p(t)\Big) \tag{17}$$

Using Appendix Equation (39) below the equation for the first moment of the distribution of random portfolio return over the time interval $[0,t]$ is...

$$\mathbb{E}\Big[r_p(t)\Big] = \Big[\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t\Big] - 1 \tag{18}$$

Using Appendix Equation (41) below the equation for the second moment of the distribution of random portfolio return over the time interval $[0,t]$ is...

$$\mathbb{E}\Big[r_p(t)^2\Big] = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\Big) - 2\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t + 1 \tag{19}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.

Using Equation (18) above the mean of the distribution of random portfolio return at time $t$ is...

$$\text{mean of } r_p(t) = \mathbb{E}\Big[r_p(t)\Big] = \left[\sum_{i=1}^{N} w_i(0)\Big(1 + m_i\Big)^t\right] - 1 \tag{20}$$

Using Equations (18) and (19) above and Appendix Equation (40) below the variance of the distribution of random portfolio return at time $t$ is...

$$\text{variance of } r_p(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\, w_j(0)\Big(1 + m_i\Big)^t\Big(1 + m_j\Big)^t \rho_i\, \rho_j \sqrt{v_i}\sqrt{v_j}\, t \tag{21}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.

Note that using the matrix and vector definitions in Equations (13), (14) and (15) above we can rewrite portfolio return mean and variance Equations (20) and (21) above as...

$$\text{mean of } r_p(t) = \vec{\mathbf{w}}^T \vec{\mathbf{r}} - 1 \text{ ...and... } \text{variance of } r_p(t) = \vec{\mathbf{w}}^T \mathbf{A} \vec{\mathbf{w}} \tag{22}$$

## Portfolio Value Equation

We will define the variable $m$ to be the portfolio's expected annual rate of return and the variable $v$ to be the portfolio's expected annual return variance. We want an equation for portfolio value that is in the following form...

$$P(t) = P(0)\Big(1 + m\Big)^t\Big(1 + \sqrt{v\, t}\, Z\Big) \text{ ...where... } Z \sim N\Big[0, 1\Big] \tag{23}$$

Using Equations (20) and (21) above the equations for the mean ($m$) and variance ($v$) in portfolio value Equation (23) above are...

$$m = \Big(1 + \text{mean of } r_p(t)\Big)^{\frac{1}{t}} - 1 \text{ ...and... } v = \frac{\text{variance of } r_p(t)}{t} \tag{24}$$

## The Answers To Our Hypothetical Problem

**Question 1**: What is portfolio value mean and variance at the end of year three?

Using Equations (13), (14), (15) and (16) above portfolio value mean and variance at the end of year three is...

$$\text{mean of } P(t) = 1000 \times \vec{\mathbf{w}}^T \vec{\mathbf{r}} = 1,195 \text{ ...and... } \text{variance of } P(t) = 1000^2 \times \vec{\mathbf{w}}^T \mathbf{A} \vec{\mathbf{w}} = 142,303 \tag{25}$$

**Question 2**: What is portfolio return mean and variance at the end of year three?

Using Equations (13), (14), (15) and (22) above portfolio return mean and variance at the end of year three is...

$$\text{mean of } r_p(t) = \vec{\mathbf{w}}^T \vec{\mathbf{r}} - 1 = 0.1945 \text{ ...and... } \text{variance of } r_p(t) = \vec{\mathbf{w}}^T \mathbf{A} \vec{\mathbf{w}} = 0.1423 \tag{26}$$

**Question 3**: What is the random portfolio value equation?

Using Equations (23) and (24) above and the answers to questions one and two above the parameters for the random portfolio value equation are...

$$m = \Big(1 + 0.1945\Big)^{\frac{1}{3}} - 1 = 0.0610 \text{ ...and... } v = \frac{0.1423}{3} = 0.0474 \text{ ...where... } \sqrt{v} = 0.2177 \tag{27}$$

Using Equation (27) above the equation for portfolio value is...

$$P(t) = P(0)\Big(1 + 0.0610\Big)^t\Big(1 + 0.2177 \times \sqrt{t} \times Z\Big) \tag{28}$$

# References


[1] Gary Schurman, *Normally-Distributed Asset Value,* September, 2018.

[2] Gary Schurman, *The Gaussian Copula,* September, 2010.

[3] Gary Schurman, *The Gaussian Copula - A Case Study,* September, 2018.


## Appendix

**A**. If the random variables $Z_c$ and $Z_i$ are independent and are normally-distributed with mean zero and variance one then note the following...

$$\mathbb{E}\bigg[Z_c\bigg]=\mathbb{E}\bigg[Z_i\bigg]=\mathbb{E}\bigg[Z_c Z_i\bigg]=0 \text{ ...and... } \mathbb{E}\bigg[Z_c^2\bigg]=\mathbb{E}\bigg[Z_i^2\bigg]=1 \tag{29}$$

**B**. Using Appendix Equation (29) above the solution to the following expectation is...

$$\begin{aligned}\mathbb{E}\bigg[\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\bigg]&=\rho_i\,\mathbb{E}\bigg[Z_c\bigg]+\sqrt{1-\rho_i^2}\,\mathbb{E}\bigg[Z_i\bigg]\\&=0\end{aligned} \tag{30}$$

**C**. Using Appendix Equation (29) above the solution to the following expectation is...

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\bigg)^2\bigg]&=\mathbb{E}\bigg[\rho_i^2\,Z_c^2+\bigg(1-\rho_i^2\bigg)Z_i^2\bigg]\\&=\rho_i^2\,\mathbb{E}\bigg[Z_c^2\bigg]+\bigg(1-\rho_i^2\bigg)\mathbb{E}\bigg[Z_i^2\bigg]\\&=1\end{aligned} \tag{31}$$

Note: Because $\mathbb{E}[Z_c\,Z_i]=0$ we can ignore all products that include that variable.

**D**. Using Appendix Equation (29) above the solution to the following expectation is...

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\bigg)\bigg(\rho_j Z_c+\sqrt{1-\rho_j^2}\,Z_j\bigg)\bigg]&=\mathbb{E}\bigg[\rho_i\,\rho_j\,Z_c^2\bigg]\\&=\rho_i\,\rho_j\,\mathbb{E}\bigg[Z_c^2\bigg]\\&=\rho_i\,\rho_j\end{aligned} \tag{32}$$

Note: Because $\mathbb{E}[Z_c\,Z_i]=\mathbb{E}[Z_c\,Z_j]=\mathbb{E}[Z_i\,Z_j]=0$ we can ignore all products that include those variables.

**E**. Using Equations (1) and (30) above the equation for the first moment of the distribution of random asset price at time $t$ is...

$$\begin{aligned}\mathbb{E}\bigg[A_i(t)\bigg]&=\mathbb{E}\bigg[A_i(0)\bigg(1+m_i\bigg)^t\bigg(1+\sqrt{v_i\,t}\bigg(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\bigg)\bigg)\bigg]\\&=A_i(0)\bigg(1+m_i\bigg)^t\bigg(1+\sqrt{v_i\,t}\,\mathbb{E}\bigg[\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\bigg]\bigg)\\&=A_i(0)\bigg(1+m_i\bigg)^t\end{aligned} \tag{33}$$

**F**. Using Equations (1), (30) and (31) above the equation for the second moment of the distribution of random asset price at time $t$ is...

$$\begin{aligned}
\mathbb{E}\Big[A_i(t)^2\Big] &= \mathbb{E}\Big[A_i(0)^2\Big(1+m_i\Big)^{2t}\Big(1+\sqrt{v_i\,t}\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)\Big)^2\Big] \\
&= \mathbb{E}\Big[A_i(0)^2\Big(1+m_i\Big)^{2t}\Big(1+2\sqrt{v_i\,t}\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)+v_i\,t\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)^2\Big] \\
&= A_i(0)^2\Big(1+m_i\Big)^{2t}\Big(1+2\sqrt{v_i\,t}\,\mathbb{E}\Big[\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big]+v_i\,t\,\mathbb{E}\Big[\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)^2\Big] \\
&= A_i(0)^2\Big(1+m_i\Big)^{2t}\Big(1+v_i\,t\Big)
\end{aligned} \tag{34}$$

**G**. Using Equations (1), (30), (31) and (32) above the equation for the expecion of the product of the prices of the i'th and j'th asset at time $t$ is...

$$\begin{aligned}
\mathbb{E}\Big[A_i(t)\,A_j(t)\Big] &= \mathbb{E}\Big[A_i(0)\Big(1+m_i\Big)^t\Big(1+\sqrt{v_i\,t}\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)\Big)A_j(0)\,\Big(1+m_j\Big)^t\Big(1+\sqrt{v_j\,t}\,\Big(\rho_j\,Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big)\Big]\Big) \\
&= A_i(0)A_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\,\mathbb{E}\Big[\Big(1+\sqrt{v_i\,t}\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)\Big)\Big(1+\sqrt{v_j\,t}\,\Big(\rho_j\,Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big)\Big)\Big] \\
&= A_i(0)A_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\,\mathbb{E}\Big[1+\sqrt{v_i\,t}\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)+\sqrt{v_j\,t}\,\Big(\rho_j\,Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big) \\
&+\sqrt{v_i\,t}\,\sqrt{v_j\,t}\,\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)\Big(\rho_j Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big)\Big] \\
&= A_i(0)A_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\sqrt{v_i\,t}\,\mathbb{E}\Big[\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big]+\sqrt{v_j\,t}\,\mathbb{E}\Big[\rho_j\,Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big] \\
&+\sqrt{v_i\,t}\,\sqrt{v_j\,t}\,\mathbb{E}\Big[\Big(\rho_i\,Z_c+\sqrt{1-\rho_i^2}\,Z_i\Big)\Big(\rho_j Z_c+\sqrt{1-\rho_j^2}\,Z_j\Big)\Big]\Big) \\
&= A_i(0)A_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\,\sqrt{v_i\,t}\,\sqrt{v_j\,t}\Big)
\end{aligned} \tag{35}$$

Note: When $i=j$ in the equation above then $\rho_i=\rho_j=1$.

**H**. Using Equations (1), (7) and (8) above the equation for the first moment of the distribution of random portfolio value at time $t$ is...

$$\begin{aligned}
\mathbb{E}\Big[P(t)\Big] &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i(t)\Big] \\
&= \sum_{i=1}^{N}\mathbb{E}\Big[A_i(t)\Big] \\
&= \sum_{i=1}^{N} A_i(0)\Big(1+m_i\Big)^t \\
&= \sum_{i=1}^{N} w_i(0)\,P(0)\Big(1+m_i\Big)^t \\
&= P(0)\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t
\end{aligned} \tag{36}$$

Note that the square of Equation (36) above is...

$$\Big[\mathbb{E}\Big[P(t)\Big]\Big]^2 = P(0)^2\sum_{i=1}^{N}\sum_{i=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t \tag{37}$$

**I**. Using Equations (1), (7), (8) and (35) above the equation for the second moment of the distribution of random portfolio value at time $t$ is...

$$\begin{aligned}
\mathbb{E}\Big[P(t)^2\Big] &= \mathbb{E}\Bigg[\sum_{i=1}^{N}\sum_{j=1}^{N} A_i(t)\,A_j(t)\Bigg] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{E}\Big[A_i(t)\,A_j(t)\Big] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} A_i(0)A_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\Big) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,P(0)\,w_j(0)\,P(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\Big) \\
&= P(0)^2\sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\Big) \qquad (38)
\end{aligned}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.

**J**. Using Equations (17) and (36) above the equation for the first moment of the distribution of random portfolio return over the time interval $[0, t]$ is...

$$\begin{aligned}
\mathbb{E}\Big[r_p(t)\Big] &= \mathbb{E}\Bigg[\frac{P(t)}{P(0)} - 1\Bigg] \\
&= \frac{1}{P(0)}\,\mathbb{E}\Big[P(t)\Big] - 1 \\
&= \Bigg[\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t\Bigg] - 1 \qquad (39)
\end{aligned}$$

Note that the square of Equation (39) above is...

$$\Bigg[\mathbb{E}\Big[r_p(t)\Big]\Bigg]^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t - 2\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t + 1 \qquad (40)$$

**K**. Using Equations (17), (36) and (38) above the equation for the second moment of the distribution of the random rate of return over the time interval $[0, t]$ is...

$$\begin{aligned}
\mathbb{E}\Big[r_p(t)^2\Big] &= \mathbb{E}\Bigg[\bigg(\frac{P(t)}{P(0)} - 1\bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\frac{P(t)^2}{P(0)^2} - 2\,\frac{P(t)}{P(0)} + 1\Bigg] \\
&= \frac{1}{P(0)^2}\,\mathbb{E}\Big[P(t)^2\Big] - \frac{2}{P(0)}\,\mathbb{E}\Big[P(t)\Big] + 1 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(0)\,w_j(0)\Big(1+m_i\Big)^t\Big(1+m_j\Big)^t\Big(1+\rho_i\,\rho_j\sqrt{v_i}\sqrt{v_j}\,t\Big) - 2\sum_{i=1}^{N} w_i(0)\Big(1+m_i\Big)^t + 1 \qquad (41)
\end{aligned}$$

Note: When $i = j$ in the equation above then $\rho_i = \rho_j = 1$.